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PLASTIC EXTENSION OF A SHEAR CRACK AT A CIRCULAR INCLUSION

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Abstract—The plastic relaxation of a shear crack situated normal to the interface of a second phase particle of circular cross-section is quantitatively analyzed. The ratio of applied stress to yield stress and the relative displacement of the crack faces at the tips of the crack in the matrix and the interface in the second phase are related to the crack parameters namely the length of the crack, the width of the plastic zone in the matrix and the second phase. The effect of the shear modulus and size of the second phase particle on the behaviour of the plastic zones is determined. A critical value of the relative displacement of the crack faces at the tip of the crack is used as the criterion to determine the tendency to brittle crack extension into the matrix and the second phase.

INTRODUCTION

The stress required to extend in a brittle manner a shear crack situated at a circular inclusion in an infinite matrix has been obtained from the analysis of pile-ups of screw dislocations [1, 2]. The plastic relaxation of a shear crack at a circular inclusion has also been analyzed [3] by the method of Atkinson and Kay [4]. The analysis assumed that the plastic zone forms at the crack tip in the matrix only so that the shear crack and the plastic zone are situated in the matrix on a plane normal to the coherent circular interface. But it has been shown [1] that a shear crack at a circular inclusion introduces internal stresses inside the inclusion which become singular at the interface. These large internal stresses inside the second phase should be relieved by the formation of another plastic zone ahead of the crack tip at the interface in the second phase. This paper takes into consideration the simultaneous formation of the plastic zones at the crack tips in the matrix and the second phase. The shear crack normal to the circular interface is represented by a double pile-up of screw dislocations and the plastic zones at the crack tips by two giant screw dislocations. The positions of the two giant screw dislocations give the extent to which the plastic zones are spread and the magnitude of their Burgers vectors gives the relative displacement of the crack faces at the tips. The results of the analysis are used to discuss the plastic relaxation of a shear crack at a circular inclusion.

ANALYSIS

Consider as shown in Fig. 1, a shear crack normal to the interface of a circular inclusion represented by a double pile-up of screw dislocations. The plastic zones at the crack tips in the matrix and the second phase are represented by giant screw dislocations of Burgers vector M_1b and M_2b respectively. The equilibrium of the shear crack with the plastic zones under the action of a stress $\sigma_{yz} = \sigma$ applied at infinity is given by

$$\int_R^{(L+R)} \frac{f(t)\,dt}{x-t} + K \int_R^{(L+R)} \frac{f(t)\,dt}{x - R^2/t} - \frac{K(N + M_1 + M_2)}{x}$$

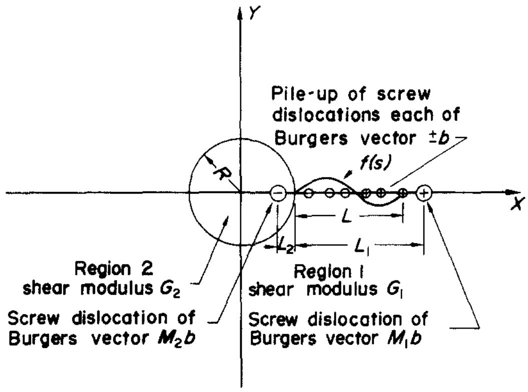

Fig. 1. Schematic illustration of a shear crack with plastic zones situated normal to the interface of a second phase particle of circular cross-section represented by a double pile-up of screw dislocations. The plastic zones at the tips in the matrix and the second phase are represented by giant screw dislocations of Burgers vector M_1b and M_2b respectively. The crack is assumed to be present before the stress is applied.

$$+\frac{M_1}{x-(L_1+R)}+\frac{KM_1}{x-R^2/(L_1+R)}+\frac{(1+K)M_2}{x-L_2}-\frac{2\pi\sigma_m}{G_1b}=0,\quad L_1>L,\ L_2<R \tag{1}$$

where $K=(G_2-G_1)/(G_2+G_1)$, $f(t)$ is the unknown dislocation distribution function representing the shear crack, N is the algebraic sum of all the dislocations in the pile-up and $\sigma_m=\sigma(1-KR^2/x^2)$ is the effective applied stress acting on the plane of the crack in the matrix [5]. Substituting $x/L=r$, $t/L=s$, $L/R+1=\beta$, $L_1/R+1=\beta_1$ and $L_2/R=\beta_2$, equation (1) can be written as

$$\int_1^{\beta}\frac{f(s)\,\mathrm{d}s}{r-s}+K\int_1^{\beta}\frac{f(s)\,\mathrm{d}s}{r-1/s}-\frac{K(N+M_1+M_2)}{r}+\frac{M_1}{r-\beta_1}+\frac{KM_1}{r-1/\beta_1}$$
$$+\frac{(1+K)M_2}{r-\beta_2}-\frac{2\pi\sigma_mR}{G_1b}=0\text{ for }\beta_1>\beta,\ \beta_2<1. \tag{2}$$

The solution to equation (2) with bounded end points is found to be

$$\begin{aligned}f(s)=&[2(m^2\cosh^2u+1)\{A_1\sinh(wu)/(p_1^2\cosh^2u-1)\\&+[A_2\sinh(wu)+A_3\cosh(gu)\sinh u]/(p_2^2\cosh^2u-1)\}\\&+4m\cosh u\{[A_4\sinh(gu)+A_5\cosh(wu)\sinh u]/(p_1^2\cosh^2u-1)\\&+A_6\sinh(gu)/(p_2^2\cosh^2u-1)\}]/(m\cosh u+1)^2\text{ for }\beta_1>1,\ 1/\beta<\beta_2<1\end{aligned} \tag{3}$$

where $\cos(w\pi)=-\cos(g\pi)=K$, $m=(\beta+1)/(\beta-1)$, $m_1=(\beta_1+1)/(\beta_1-1)$, $m_2=(\beta_2+1)/(1-\beta_2)$, $p_1=m/m_1$, $p_2=m/m_2$ and $u=\cosh^{-1}\{(s+1)/m(s-1)\}$. The constants A_1 to A_6 are determined by substituting for $f(s)$ in equation (3) and evaluating the integrals using contour integration. Further, equating the constant terms, the coefficients of $1/r$, $1/r^2$, $1/(r-\beta_1)$, $1/(r-1/\beta_1)$ and $1/(r-\beta_2)$ on both sides of equation (2) and solving the resulting equations gives the constants A_1 to A_6. The above solution to equation (2) is valid for all values of $\beta_1>1$ and $1/\beta<\beta_2<1$. Thus the results obtained will be valid for all widths of the plastic zone in the matrix but the width of the plastic zone in the second phase is restricted to extend upto a point $1/\beta$ from the origin in the second phase. As pointed out by Atkinson and Kay [4], the model used here in the analysis of the plastic zone at the crack tips is more valid for small scale yielding and therefore the solution given by equation (3) suffices. The boundedness condition [6] of the distribution function is applied by writing equation (2) in the form

$$\int_1^\beta \frac{f(s)\,\mathrm{d}s}{r-s} = \frac{2\pi R\sigma_m}{G_1 b} + \frac{K(N+M_1+M_2)}{r} - \frac{M_1}{r-\beta_1} - \frac{KM_1}{r-1/\beta_1} - \frac{(1+K)M_2}{r-\beta_2} - K\int_1^\beta \frac{f(s)\,\mathrm{d}s}{r-1/s} = g(r). \tag{4}$$

The integral on the right hand side is evaluated by substituting for $f(s)$ from equation (3) so that the above equation becomes an equation with a simple Cauchy type integral on the left and a function of r on the right. The boundedness condition of the distribution function can now be applied in the form

$$\int_1^\beta \frac{g(r)\,\mathrm{d}r}{[(r-1)(\beta-r)]^{1/2}} = 0. \tag{5}$$

It may be noted that equation (5) relates the applied stress, the Burgers vectors of the giant screw dislocations M_1b and M_2b to the crack parameters β, β_1 and β_2. Two more equations to determine the applied stress, M_1b and M_2b in terms of the crack parameters are obtained by invoking the equilibrium of the two giant screw dislocations. The equilibrium of the giant screw dislocation at $(L_1, 0)$ is given by the condition that the total stress acting on it should become zero. Thus, assuming the frictional stress opposing the movement of the giant screw dislocation in the plastic zone in the matrix to be equal to the yield stress σ_y of the matrix, the condition of zero stress gives,

$$\int_1^\beta \frac{f(s)\,\mathrm{d}s}{\beta_1-s} + K\int_1^\beta \frac{f(s)\,\mathrm{d}s}{\beta_1-1/s} - \frac{K(N+M_1+M_2)}{\beta_1} + \frac{KM_1\beta_1}{\beta_1^2-1} + \frac{(1+K)M_2}{\beta_1-\beta_2} - \frac{2\pi R(\sigma_m-\sigma_y)}{G_1 b} = 0. \tag{6}$$

Similarly the equilibrium of the giant screw dislocation inside the second phase at $(L_2, 0)$ can be obtained using the σ_{yz} component of stress field due to a screw dislocation in the second phase [5]. Assuming the frictional stress opposing the movement of the giant dislocation in the plastic zone in the second phase to be equal to the yield stress of the second phase, the condition of zero stress gives

$$(1+K)\int_1^\beta \frac{f(s)\,\mathrm{d}s}{\beta_2-s} + \frac{(1+K)M_1}{\beta_2-\beta_1} - \frac{K(1+K)M_2\beta_2}{(1-K)(\beta_2^2-1)} - \frac{2\pi R(\sigma_i-\sigma_y)}{G_1 b} = 0 \tag{7}$$

where $\sigma_i = \sigma(1-K)$ is the effective applied stress acting on the $y=0$ plane in the second phase [5]. In equation (7), it is assumed for convenience that the yield stress of the matrix and the second phase are equal. σ/σ_y, $G_1M_1b/2\pi L_1\sigma_y$ and $G_1M_2b/2\pi L_2\sigma_y$ are determined explicitly in terms of the crack parameters β, β_1 and β_2 by solving equations (5–7). The lengthy mathematical expressions involved in these equations are avoided here to conserve space but interested readers can obtain the exact details from the author.

DISCUSSION

The results obtained in the previous section may now be analyzed numerically to determine the behaviour of the plastic zones at the crack tips. σ/σ_y is the ratio of applied stress to yield stress. $G_1M_1b/2\pi L_1\sigma_y$ is considered as the relative displacement of crack faces at the tip of the crack in the matrix (RDC 1). Similarly $G_1M_2B/2\pi L_2\sigma_y$ is considered as the relative displacement of crack faces at the tip of the crack in the second phase (RDC 2). β_1 and β_2 give the extent to which the plastic zones in the matrix and the second phase respectively are spread under the action of the applied stress at infinity. As pointed earlier, the results of the analysis are expected to be more valid for small scale yielding, i.e. for β/β_1 near unity and β_2/β near zero. Also values of σ/σ_y less than unity only are relevant. σ/σ_y, RDC 1 and RDC 2 are shown in Figs. 2 and 3 as functions of β/β_1 for $\beta = 2$, $K = -0{\cdot}5$ and $0{\cdot}5$ and various values of the ratio of the width of the

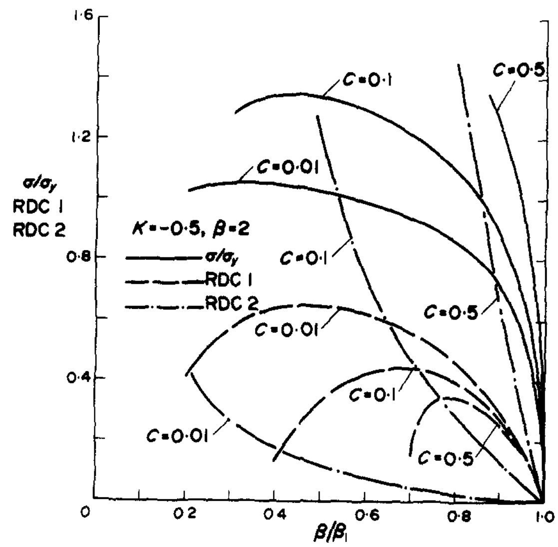

Fig. 2. σ/σ_y, RDC 1 and RDC 2 shown against β/β_1 for $K = -0{\cdot}5$, $\beta = 2$ and various values of C.

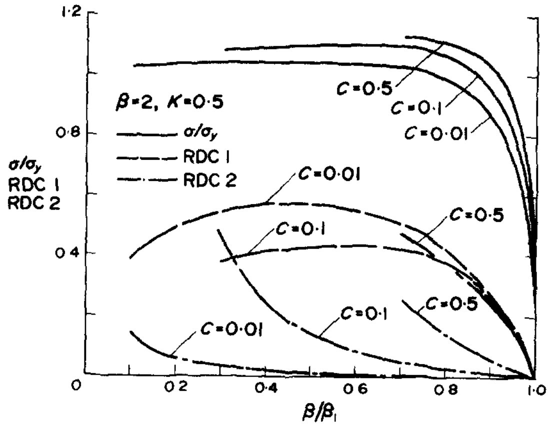

Fig. 3. σ/σ_y, RDC 1 and RDC 2 shown against β/β_1 for $K = 0{\cdot}5$, $\beta = 2$ and various values of C.

plastic zone in the second phase to the width of the plastic zone in the matrix, i.e. $C = (1-\beta_2)/(\beta_1-\beta)$. These curves are plotted against β/β_1 such that for a given value of C, the condition imposed by the solution namely $\beta_2 > 1/\beta$ is satisfied. Thus the range of β/β_1 for any β decreases with increasing values of C. It is seen that for small scale yielding (β/β_1 near unity), σ/σ_y and RDC 1 increase with increasing width of the plastic zone in the matrix (β/β_1 decreasing) and the second phase (β/β_2 decreasing). But σ/σ_y and RDC 1 decrease with increasing degree of yielding for large scale yielding. RDC 2 increases with increasing degree of yielding in the complete range. For a given width of the plastic zone in the matrix, σ/σ_y and RDC 2 increase but RDC 1 decreases with increasing width of the plastic zone in the second phase. A critical RDC criterion (CRDCC) may now be used to predict the tendency to brittle crack extension namely if the relative displacement of the crack faces at the tip exceeds certain critical value, the crack extends in a brittle manner. Application of critical RDC criterion (CRDCC) to RDC 1 and RDC 2 in Figs. 2 and 3 indicates that the tendency to brittle crack extension into the matrix decreases with increasing values of C but the opposite is true for brittle extension into the second phase. In order to show the effect the shear modulus of the second phase, σ/σ_y, RDC 1 and RDC 2 are shown in Fig. 4 as functions of β/β_1 for $C = 0{\cdot}05$, $\beta = 2$ and various values of K. σ/σ_y increases but RDC 1 decreases with increasing hardness of the second phase particle for small scale yielding. The opposite happens for large scale yielding. RDC 2 decreases with increasing values of K in the complete range of yielding. Therefore the tendency to brittle crack extension into the matrix and the second phase decreases with increasing hardness of the second phase for small scale yielding. The effect of the particle size on the behaviour of the plastic zones is shown in Fig. 5 for $C = 0{\cdot}05$ and $K = 0{\cdot}3$. σ/σ_y decreases and RDC 2 increases with decreasing particle size. RDC 1 decreases with decreasing particle size for small scale yielding but the opposite happens for large scale yielding. These results are also found to be valid for any negative value of K. The effective applied stress σ_m on the giant screw dislocation increases with decreasing particle size for a positive value of K. But the image repulsive force due to the second phase on the giant dislocation decreases. The net result is a less rapid decrease in σ/σ_y with decreasing particle size. Application of CRDCC to RDC 1 and RDC 2 shows that the tendency to brittle extension into the matrix decreases but the tendency to brittle extension into the second phase increases with decreasing particle size.

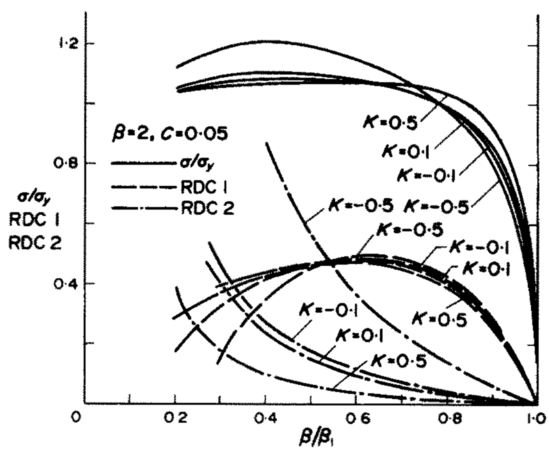

Fig. 4. σ/σ_y, RDC 1 and RDC 2 shown against β/β_1 for $\beta = 2$, $C = 0{\cdot}05$ and various values of K.

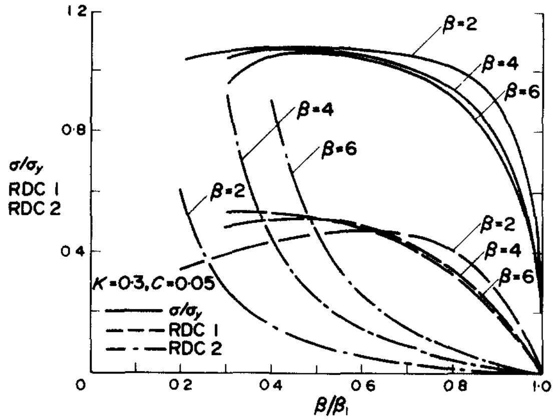

Fig. 5. σ/σ_y, RDC 1 and RDC 2 shown against β/β_1 for $C = 0{\cdot}05$, $K = 0{\cdot}3$ and various values of β.

SUMMARY AND CONCLUSIONS

The plastic relaxation of a shear crack situated normal to the interface of a second phase particle of circular cross-section is quantitatively analyzed and the behaviour of the plastic zones at the tips in the matrix and the second phase obtained. The following important conclusions have been arrived at.

(1) The ratio of applied stress to yield stress (σ/σ_y) and the relative displacement of crack faces at the tip of the crack in the matrix (RDC 1) and the second phase (RDC 2) increase with increasing width of the plastic zone in the matrix for small scale yielding. For a given width of the plastic zone in the matrix, σ/σ_y and RDC 2 increase but RDC 1 decreases with increasing width of the plastic zone in the second phase. Application of critical RDC criterion (CRDCC) shows that with increasing width of the plastic zone in the second phase the tendency to brittle crack extension into the matrix decreases but the tendency to brittle extension into the second phase increases.

(2) σ/σ_y is higher for higher values of the shear modulus of the second phase particle and small scale yielding. This is not true for large scale yielding. The tendency to brittle crack extension into the matrix decreases with increasing shear modulus of the second phase for small scale yielding only. The tendency to brittle crack extension into the second phase decreases with increasing shear modulus of the second phase in the complete range of yielding.

(3) σ/σ_y decreases with decreasing particle size for all values of the shear modulus of the second phase. The tendency to brittle crack extension into the matrix decreases with decreasing particle size for small scale yielding only. The tendency to brittle crack extension into the second phase increases with decreasing particle size in the complete range of yielding.

It should be pointed out finally that as a consequence of the model used in the analysis of the plastic zones the results obtained in this paper are more correct for small scale yielding.

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